Dynamic Meteorology

Lecture - 01

[Tuesday, 2022-01-18]

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…Few messages beforehand…

- Brush up our the concepts of 12th standard mathematics and physics e.g. the equations, functions, graphs, differentiation, integration concepts and formulae. It would be handy!
- During the live class you are suggested to make your notes (pen/paper) in parallel.
- Course requirements (e.g. Attendance, Theory & Practicals) must be fulfilled.

References:

- *An Introduction to Dynamic Meteorology [by James Holton]*
- *Atmospheric Science: An Introductory Survey [by Wallace and Hobbs]*
- *Meteorology for Scientists and Engineers [by Roland Stull]*
- *An Introduction to Atmospheric Physics [by David Andrews]*

Geophysical Fluid Dynamics

- *Geophysical fluid dynamics (GFD) is the study of the dynamics of the fluid systems of earth and planets. The principal fluid systems we are interested are the atmosphere and oceans.*
- *The basis of GFD lies in the principles of conservation of momentum, mass and energy. These are expressed mathematically in Newton's equations of motion for a continuous medium, the equation of continuity, and the thermodynamic energy equation.*

- Two main ingredients distinguish the GFD from traditional fluid mechanics:
  - The effects of Rotation and those of Stratification.
  - The controlling influence of one, the other, or both leads to peculiarities exhibited only by geophysical flows.

Governing laws

- Atmospheric/Oceanic motions are governed by three principals:
  *conservation of momentum
  *conservation of mass
  *conservation of energy
- There are two approaches to describe the motion of a fluid and its associated properties: **Eulerian** and **Lagrangian**.
- These conservation laws can be applied to a control volume of the atmosphere at a fixed location (Eulerian) or to a control volume of the atmosphere that is moving with the flow (Lagrangian – refers to as the 'material volume').
- **Lagrangian:** We deal with the total, substantial or material derivative.
- **Eulerian:** We deal with the local or partial derivative.

Fluid motions

Two Ways to Describe Fluid Flow

- ***Eulerian:* *Stay put and watch the flow***
- ***Lagrangian:* *Drift along, see where you go.***

The independent variables are the space and time coordinates, $\mathbf{r} = (x, y, z)$ and t.

The dependent variables are the velocity, pressure, density and temperature, $\mathbf{V} = (u, v, w)$, p, ρ and T.

Further variables are needed for a fuller treatment, e.g. humidity q in the atmosphere and salinity s in the ocean.

Each variable is a function of both position and time.

For example,

$$p = p(x, y, z, t)$$

Fluid motions

We must consider variations with respect to space and time.

$$p = p(x, y, z, t) .$$

- ***Eulerian:* *Stay put and watch the flow***

We denote the change of pressure with time at a fixed point by the Eulerian (or partial) derivative:

$$\frac{\partial p}{\partial t} \qquad x, y \text{ and } z \text{ fixed.}$$

- ***Lagrangian:* *Drift along, see where you go.***

We denote the change of pressure with time following the flow by the Lagrangian (or material or total) derivative:

$$\frac{dp}{dt} \qquad \text{parcel of fluid fixed.}$$

Fluid motions

Connection: $\partial p/\partial t \Longleftrightarrow dp/dt$

The pressure is a function of both space and time:

$$p = p(x(t), y(t), z(t), t) \,.$$

The total variation, following the flow, is given by the chain rule:

$$\begin{aligned}\frac{dp}{dt} &= \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial p}{\partial z}\cdot\frac{dz}{dt}\\ &= \frac{\partial p}{\partial t} + u\cdot\frac{\partial p}{\partial x} + v\cdot\frac{\partial p}{\partial y} + w\cdot\frac{\partial p}{\partial z}\\ &= \frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p \,.\end{aligned}$$

This is true for all variables, so we have

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{V}\cdot\nabla(\ )\,.$$

Fluid motions

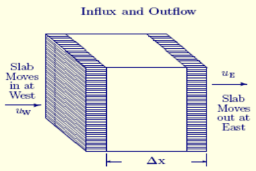
Conservation of Mass

Air is neither created nor destroyed. Therefore, the total mass must remain constant. Moreover, the mass of an identifiable parcel of air must remain unchanged with time.

The mathematical expression of mass conservation is the Continuity Equation.

Fluid motions

Eulerian Formulation



Consider a cubic region of dimensions $\Delta x = \Delta y = \Delta z$, fixed in space.

Air flows freely through the region.

The change of mass of the air in the cube must equal the net flux of mass into or out of the region.

For simplicity, consider flow in the x-direction. Let u_W be the x-component of velocity at the western face, and u_E be the x-component of velocity at the eastern face.

Fluid motions

Total mass of air in the box (density × volume):

$$M = \rho\cdot\Delta x\Delta y\Delta z = \rho\mathcal{V}$$

Change of mass in time Δt (volume is fixed):

$$\Delta M = \frac{\partial M}{\partial t}\Delta t = \frac{\partial \rho}{\partial t}\Delta t\cdot\mathcal{V}\,.$$

Influx at western face (density × slab volume):

$$\rho_W(u_W\Delta t)\Delta y\Delta z = (\rho u)_W\Delta t\cdot\Delta y\Delta z$$

Outflow at eastern face (density × slab volume):

$$\rho_E(u_E\Delta t)\Delta y\Delta z = (\rho u)_E\Delta t\cdot\Delta y\Delta z$$

Net flow $\mathcal{F}$ into the box (influx − outflow) in time Δt:

$$\mathcal{F} = [(\rho u)_W - (\rho u)_E]\Delta t\cdot\Delta y\Delta z = -\frac{(\rho u)_E - (\rho u)_W}{\Delta x}\Delta t\cdot\Delta x\Delta y\Delta z$$

But $\Delta M = \mathcal{F}$, so the quantities in red must be equal:

$$\frac{\partial\rho}{\partial t} = -\frac{(\rho u)_E - (\rho u)_W}{\Delta x} \approx -\frac{\partial(\rho u)}{\partial x}$$

Fluid motions

Thus, for flow only in the x-direction we have

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

However, there is also flow through the front and back faces, and through the top and bottom of the box.

Symmetry arguments lead us immediately to the result

$$\frac{\partial\rho}{\partial t} = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)$$

This may be written using the divergence operator as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0\,.$$

This is the Eulerian form of the continuity equation. It is one of the fundamental equations of atmospheric dynamics.

End of Lecture - 01

Dynamic Meteorology

Lecture - 02

[Thursday, 2022-01-20]

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Fluid motions

$$\underbrace{\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0}_{\text{Lagrangian Form}} \qquad \underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0}_{\text{Eulerian Form}}$$

We recall the relationship between the time derivatives:

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{V}\cdot\nabla(\ )\,.$$

We also note the vector identity

$$\nabla\cdot\rho\mathbf{V} = \mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V}\,.$$

Substituting in the Lagrangian form, we get:

$$\begin{aligned}\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} &= \left(\frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla\rho\right) + \rho\nabla\cdot\mathbf{V}\\ &= \frac{\partial\rho}{\partial t} + \left(\mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V}\right)\\ &= \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0\,.\end{aligned}$$

Thus the equivalence of the two forms is established. QED

Fluid motions

Incompressibility

For an incompressible fluid, the volume of a parcel remains unchanged. Thus, the material density is constant following the flow: $d\rho/dt = 0$. Thus, the continuity equation reduces to

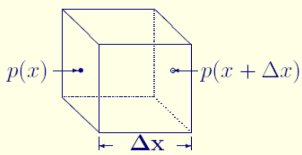
$$\nabla\cdot\mathbf{V} = 0\,.$$

The assumption of incompressibility is a natural one for the ocean. For the atmosphere, it is less obviously reasonable. Indeed, many atmospheric phenomena depend on compressibility. However, the essential large scale dynamics can be successfully modelled by an incompressible fluid.
The benefit of assuming incompressibility is that we get a closed system without having to consider the thermodynamics explicitly. For compressible flow, we would have to have another equation for ρ, the thermodynamic equation. But this introduces the temperature T, and yet another equation, the equation of state, is required.

Forces on an Air Parcel

Pressure Force

Pressure on Box



Consider a cubic box of air, of dimension $\Delta x \times \Delta y \times \Delta z = \mathcal{V}$.

The pressure acts *normally* on each face of the cube.

Net force on left-hand face:

$$p(x)\cdot\Delta y\Delta z$$

Net force on right-hand face:

$$-p(x+\Delta x)\cdot\Delta y\Delta z$$

Total pressure force in the x-direction:

$$-\Big[p(x+\Delta x) - p(x)\Big]\cdot\Delta y\Delta z$$

Forces on an Air Parcel

Total pressure force in x-direction:

$$-\Big[p(x+\Delta x) - p(x)\Big]\cdot\Delta y\Delta z = -\left(\frac{p(x+\Delta x) - p(x)}{\Delta x}\right)\cdot\Delta x\Delta y\Delta z$$

But $\Delta x\Delta y\Delta z = \mathcal{V}$, so the force <u>per unit volume</u> is:

$$-\left(\frac{p(x+\Delta x) - p(x)}{\Delta x}\right) \approx -\frac{\partial p}{\partial x}$$

A parcel of mass m has volume $\mathcal{V} = m/\rho$, so a unit mass has volume $1/\rho$. The pressure force <u>per unit mass</u> in the x-direction is thus

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similar arguments apply in the y and z directions. So, the vector force <u>per unit mass</u> due to pressure is

$$\mathbf{F}_p = \left(-\frac{1}{\rho}\frac{\partial p}{\partial x}, -\frac{1}{\rho}\frac{\partial p}{\partial y}, -\frac{1}{\rho}\frac{\partial p}{\partial z}\right) = -\frac{1}{\rho}\nabla p\,.$$

This force acts in the direction of lower pressure.

Forces on an Air Parcel

Force of Gravity

Newton's law of gravity states that two bodies of mass m_1 and m_2 attract each-other with a force given by

$$F = G\frac{m_1 m_2}{d^2}$$

where d the distance between them. The constant G is the universal gravitational constant.

Near the earth's surface, a parcel of air of mass m is attracted towards the earth with a force

$$F = m\frac{GM}{a^2}$$

where M is the mass of the earth and a its radius.

We define the <u>acceleration due to gravity</u> by

$$g = \frac{GM}{a^2}$$

The acceleration due to gravity can be evaluated as follows:

$G = 6.672\times10^{-11}$, $M = 5.974\times10^{24}$, $a = 6.375\times10^{6}$ $\Longrightarrow$ $g = 9.807$

(all values are in SI units). So, roughly, $g \approx 10\,\mathrm{m\,s^{-2}}$.

The force due to gravity acts vertically downward, towards the centre of the earth. If k is a unit vector pointing upward, we may write it:

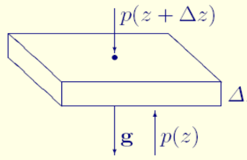
$$\mathbf{F}_g = -mg\mathbf{k}\,.$$

Forces on an Air Parcel

Hydrostatic Balance

For a fluid at rest, the pressure at a point depends on the weight of fluid vertically above that point.

The pressure difference between two points on the same vertical line depends only on the weight of fluid between them.



Force Upward on Box : $+[p(z)\cdot\Delta x\Delta y]$

Force Downward on Box : $-[p(z+\Delta z)\cdot\Delta x\Delta y + mg]$

Forces on an Air Parcel

For equilibrium, the net force must be zero:

$$\frac{p(z+\Delta z)-p(z)}{\Delta z}\cdot\Delta x\Delta y\Delta z + mg = 0\,.$$

This may be written

$$\frac{\partial p}{\partial z}\cdot\mathcal{V} + mg = 0\,,$$

or, dividing through by the volume,

$$\frac{\partial p}{\partial z} + \rho g = 0\,.$$

This is the Hydrostatic balance equation. It implies an exact balance between the vertical pressure gradient and gravity.

For an atmosphere at rest, hydrostatic balance holds exactly.

Reference frames

- *Inertial reference frame:*
  - *Reference frame at rest or moving at constant velocity, such as one fixed in space.*
- *Non-inertial reference frame*
  - *Reference frame accelerating or rotating, such as an object at rest on earth or in motion relative to the earth.*
- *True force*
  - *Force that exists when an observation is made from an inertial frame (e.g., gravitational force, pressure-gradient force, viscous force) – Does work.*
- *Apparent (virtual or inertial) force*
  - *Fictitious force that appears to exist when an observation is made from an non-inertial reference frame, but is an acceleration from an inertial reference frame (e.g., centrifugal force, Coriolis force) – Does no work.*

Rotating coordinate system

- **Newton's second law** can be used to derive an equation that describes conservation of momentum (one of the basic principles of atmospheric dynamics), but this law applies to motion in an inertial reference frame.

- In order to apply this law in a non-inertial reference frame, we either need to consider apparent forces that arise due to the motion of the non-inertial reference frame, or we need to relate the acceleration vector in an inertial reference frame to the acceleration vector in a non-inertial reference frame.

Applying this concept to a position vector $\vec{\mathbf{r}}$

$$\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{Inertial} = \left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{Rotating} + \vec{\Omega}\times\vec{\mathbf{r}} \quad\Rightarrow\quad \vec{\mathbf{V}}_{\mathbf{I}} = \vec{\mathbf{V}}_{\mathbf{R}} + \vec{\Omega}\times\vec{\mathbf{r}}$$

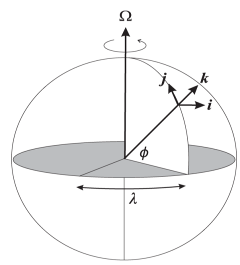
$$\left(\frac{d\vec{\mathbf{V}}}{dt}\right)_I = \left(\frac{d\vec{\mathbf{V}}}{dt}\right)_R + 2\vec{\Omega}\times\vec{\mathbf{V}} + \Omega^2\vec{\mathbf{R}}$$

(more details, Holton, Pedlosky)

With Newton's second law of motion, $\sum_i \vec{\mathbf{F}}_i = \left(\frac{d\vec{\mathbf{V}}}{dt}\right)_I$

$$\left(\frac{d\vec{\mathbf{V}}}{dt}\right)_R + \left(2\vec{\Omega}\times\vec{\mathbf{V}} + \Omega^2\vec{\mathbf{R}}\right)_{apparent} = \sum_i \vec{\mathbf{F}}_i$$

where the right hand terms represent the summation of real (true) forces, and the apparent forces are added to represent the equations of motion in an inertial frame of reference.



Effective gravity

- Newton's second law can be used to derive an equation that describes conservation of momentum (one of the basic principles of atmospheric dynamics), but this law applies to motion in an inertial reference frame.

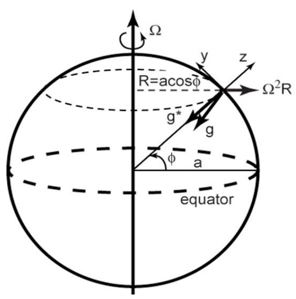
- In order to apply this law in a non-inertial reference frame, we either need to consider apparent forces that arise due to the motion of the non-inertial reference frame or we need to relate the acceleration vector in an inertial reference frame to the acceleration vector in a non-inertial reference frame.

Effective gravity

The centrifugal force is a body force and can be combined with gravity force to give an effective gravity $\vec{\mathbf{g}} = -g\hat{\mathbf{k}} = \vec{\mathbf{g}}^* + \Omega^2\vec{\mathbf{R}}$.

Gravity can be represented as a gradient of a scalar potential function known as geopotential (Φ).

$$\vec{\mathbf{g}} = -g\hat{\mathbf{k}} = -\nabla\Phi \quad\Rightarrow\quad d\Phi = g dz$$



Equations of motion in a rotating coordinate frame

- Rate of change of absolute velocity following the motion = Sum of all forces acting per unit mass

$$\frac{d\vec{U}}{dt} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\vec{\nabla}P + \vec{g} + \vec{F}_r$$

$-2\vec{\Omega}\times\vec{U}$ → **Coriolis force**

$-\frac{1}{\rho}\vec{\nabla}P$ → **Pressure gradient force**

$\vec{g} = \vec{g}^* + \Omega^2\vec{R}$ → **Gravity force per unit mass = Sum of the gravitational and centrifugal force terms**

$\vec{F}_r$ → **Frictional force in the fluid**

Note: The Coriolis and Centrifugal force terms are apparent / fictitious forces

Momentum equations in spherical coordinate system

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega(v\sin\varphi - w\cos\varphi) - \frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho a}\frac{\partial p}{\partial\varphi}$$

$$\frac{dw}{dt} - \frac{(u^2+v^2)}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{d}{dt}(\ ) \equiv \left[\frac{\partial}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}\right](\ )$$

Real force Initiates the motion

Nonlinear

These are the curvature terms that arise because the tangential coordinate system curves with the surface of the Earth.

These are the Coriolis terms that arise because the tangential coordinate system rotates with the Earth.

Complete set of Equations

✓ The complete set of model equations when moisture (q = specific humidity) is included contains seven unknowns: U (u, v, w); T; p (ρ or α) and q

✓ This forms seven equations:

$\frac{d\vec{U}}{dt} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\vec{\nabla}P + \vec{g} + \vec{F}_r$ → **3 Momentum equations for the 3 components (u, v, w)**

$\frac{\partial\rho}{\partial t} + \vec{u}.\nabla\rho + \rho\nabla.\vec{u} = 0$ → **Continuity equation**

$C_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{Q}$ → **Thermodynamic energy equation**

$p\alpha = RT$ → **Equation of state**
α = Inverse of density = Specific volume

$\frac{\partial\rho q}{\partial t} = -\nabla.\left(\rho\vec{U}q\right) + \rho(E - C)$ → **Moisture equation**
E = Evaporation (moisture source)
C = Condensation (moisture sink)

Scale Analysis

- **Scale Analysis**: It is a convenient technique for estimating the magnitudes of various terms in the governing equations for a particular type of motion.
- In scaling, typical expected values of the following quantities are specified:
  - (1) magnitudes of the field variables;
  - (2) amplitudes of fluctuations in the field variables; and,
  - (3) the characteristic length (L), depth (H), and time (T) scales on which these fluctuations occur.
- These typical values are then used to compare the magnitudes of various terms in the governing equations.

Scale Analysis

- For example, in a typical midlatitude synoptic (wide region) cyclone, the surface pressure might fluctuate by 10 hPa over a horizontal distance of 1000 km.
- Designating the amplitude of the horizontal pressure fluctuation by **δp**, the horizontal coordinates by **x** and **y**, and the horizontal scale by **L**, the magnitude of the horizontal pressure gradient may be estimated by dividing **δp** by the length **L** to get:

$$\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) \sim \frac{\delta p}{L} = 10\ \text{hpa}/10^3\ \text{km}\ \left(10^{-3}\ \text{Pa m}^{-1}\right)$$

- Pressure fluctuations of similar magnitudes occur in systems of vastly different scale e.g. tornadoes, squall lines, and hurricanes. Thus, the horizontal pressure gradient can range over several orders of magnitude for systems of meteorological interest.

Scale Analysis

- Similar considerations are also valid for derivative terms involving other field variables.
- Therefore, the nature of the dominant terms in the governing equations is crucially dependent on the horizontal scale of the motions.
- In particular, motions with horizontal scales of a few kilometers or less tend to have short time scales so that terms involving the rotation of the earth are negligible, while for larger scales they become very important.
- Because the character of atmospheric motions depends so strongly on the horizontal scale, this scale provides a convenient method for the classification of motion systems.

Scale Analysis

- Scaling arguments are used extensively in developing simplifications of the governing equations for use in modeling various types of motion systems.
- The Table here classifies examples of scales of various types of motions in the Atmosphere and Ocean.

TABLE 1.2 Length, Velocity and Time Scales in the Earth's Atmosphere and Oceans

Phenomenon	Length Scale L	Velocity Scale U	Timescale T
Atmosphere			
Microturbulence	10–100 cm	5–50 cm/s	few seconds
Thunderstorms	few km	1–10 m/s	few hours
Sea breeze	5–50 km	1–10 m/s	6 h
Tornado	10–500 m	30–100 m/s	10–60 min
Hurricane	300–500 km	30–60 m/s	Days to weeks
Mountain waves	10–100 km	1–20 m/s	Days
Weather patterns	100–5000 km	1–50 m/s	Days to weeks
Prevailing winds	Global	5–50 m/s	Seasons to years
Climatic variations	Global	1–50 m/s	Decades and beyond
Ocean			
Microturbulence	1–100 cm	1–10 cm/s	10–100 s
Internal waves	1–20 km	0.05–0.5 m/s	Minutes to hours
Tides	Basin scale	1–100 m/s	Hours
Coastal upwelling	1–10 km	0.1–1 m/s	Several days
Fronts	1–20 km	0.5–5 m/s	Few days
Eddies	5–100 km	0.1–1 m/s	Days to weeks
Major currents	50–500 km	0.5–2 m/s	Weeks to seasons
Large-scale gyres	Basin scale	0.01–0.1 m/s	Decades and beyond

End of Lecture - 02

Dynamic Meteorology

Lecture - 03

[Tuesday, 2022-01-25]

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Continuity equation: Conservation of mass

Local increase of density with time must be balanced by divergence of mass flux $(\rho \vec{u})$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{u}) = 0$$

equivalently

$$\frac{\partial \rho}{\partial t} + \vec{u}.\nabla\rho + \rho\nabla.\vec{u} = 0$$

or

$$\frac{d\rho}{dt} + \rho \nabla . \vec{u} = 0$$

Thermodynamic energy equation

- Conservation of thermodynamic energy as applied to a moving fluid element
- According to the first law of thermodynamics, the change in the internal energy of a system is equal to the difference between the heat added to the system and the work done by the system

Dry atmosphere

$$C_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$$

- C_p ⟹ Specific heat of dry air at constant Pressure = 1004 J kg^{-1} K^{-1}
- α ⟹ Specific volume (inverse of density)
- $\dot{Q}$ ⟹ Rate of heating per unit mass (eg. conduction, radiation, convection)

Thermodynamic energy equation

Dividing the previous equation by T and using Equation of State: The first law of thermodynamics takes the Entropy form

$$C_p \frac{d(\ln T)}{dt} - R\frac{d(\ln P)}{dt} = \frac{\dot{Q}}{T} = \frac{dS}{dT}$$

R= Gas constant for dry air = 287 J kg^{-1} K^{-1}

- The above equation gives the rate of change of entropy per unit mass following the motion for a thermodynamically reversible process.

Complete set of Equations

✓ The complete set of model equations when moisture (q = specific humidity) is included contains seven unknowns: U (u, v, w); T; p (ρ or α) and q

✓ This forms seven equations:

$$\frac{d\vec{U}}{dt} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\nabla P + \vec{g} + \vec{F}_r$$ → 3 Momentum equations for the 3 components (u, v, w)

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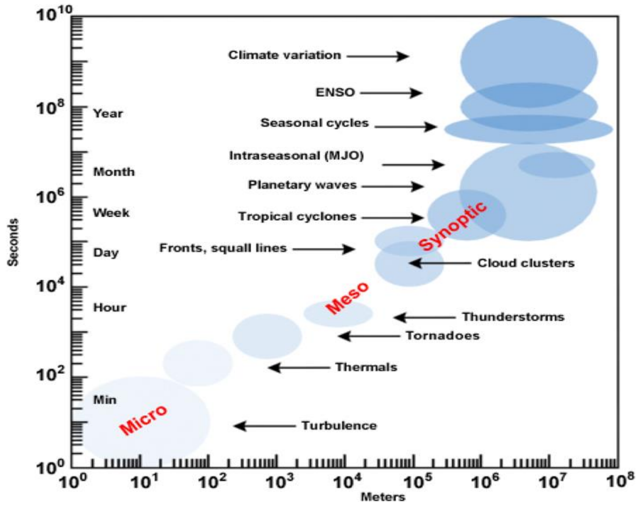
Scale of fluid motions



Scale analysis: Synoptic systems

- Large-scale, low-frequency flows of geophysical fluids (eg., atmosphere, oceans) are strongly influenced by Earth's rotation.

- Time-period for a fluid element moving with speed of "U" to traverse a length "L" should be greater than the period of the Earth's rotation – so that the fluid feels or senses the rotational effects over the time-scale of motion.

- **Rossby Number (R_o):** A non-dimensional number (ratio of the characteristic scales for the acceleration and the Coriolis force terms) estimates the importance of rotation.

$$Ro = \text{Rossby number} = \frac{\text{inertial acceleration}}{\text{Coriolis acceleration}} = \frac{U^2/L}{2\Omega U} = \frac{U}{2\Omega L} \qquad 2\Omega = 10^{-4}\ s^{-1}$$

- For large-scale flows **$R_o \le 1$**

Scale analysis: Rossby numbers

Flow system	L	U (ms^{-1})	Rossby number
Ocean circulation	1000 - 5000 km	1-10	0.01 – 0.1
Extra-tropical cyclone	1000 km	1-10	0.01 – 0.1
Tropical cyclone	500 km	50 or greater	~ 1
Tornado	100 m	100	10^4

- A typical synoptic scale motion in the atmosphere has a length scale L ~ 1000 km and velocity scale U ~ 20 ms^{-1}. This is a low Rossby number flow since **R_o** < 1 (eg. The 500 hPa height field).

- In the gulf-stream, U ~ 100 cms^{-1} and the characteristic horizontal scale L ~ 100 km. Here **R_o** ~ 0.07 and qualifies as large-scale motion.

- Consider a tornado – which has a typical tangential velocity say 30 ms^{-1} at a distance of 300 m from the center of the vortex. It turns out to be high Rossby number flow (**R_o** ~ 700).

Scale analysis: Synoptic systems

Element	Typical value	Magnitude
U, V (horizontal velocity)	10-20 m s^{-1}	10^1
W (vertical velocity)	1 cm s^{-1}	10^{-2}
L (length, distance scale)	1000 km	10^6
H (depth, height scale)	10 km (depth of troposphere)	10^4
Horizontal pressure change	10-20 hPa	10^3
Vertical pressure change	1000 hPa	10^5
Time (L/U)	27 hours	10^5
ρ (density)	1 kg m^{-3}	10^0
g (gravity)	9.8 m s^{-2}	10^1
Ω (angular velocity)	7.292×10^{-5} s^{-1}	10^{-4}

Balanced Flows

✓ If we incorporate a small number of approximations, or idealizations, we can gain a qualitative understanding of the horizontal balance of forces in the ocean and atmosphere.

✓ The hydrostatic balance (vertical), which assumes that friction and coriolis forces have negligible effects, has only a 0.01% error when applied to synoptic scale motions.

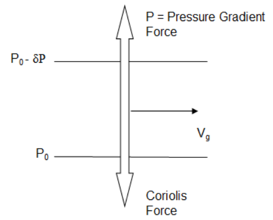
✓ Horizontal errors are somewhat larger, but still small enough to allow application to a large number of cases

Rossby Number

Small	~ 1	Large
Geostrophic Flow	Gradient Flow	Cyclostrophic Flow

Geostrophic Balance

- For mid-latitude synoptic scale disturbances, the Coriolis force and the pressure gradient force are in approximate balance.
- The geostrophic balance is a diagnostic expression which gives an approximate relationship between the pressure field and horizontal velocity in synoptic scale systems for the mid-latitudes
- Geostrophic flow is parallel to the isobars
- It is only a diagnostic relation and cannot be used to predict the evolution of the velocity field



$$2\vec{\Omega} \times \vec{U} = -\frac{1}{\rho}\vec{\nabla} P$$

$$\vec{V}_g = \hat{k} \times \frac{1}{\rho f}\vec{\nabla} P$$

$$f = 2\Omega \sin(\phi)$$ **Coriolis parameter**

$$-fv \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad fu \cong -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Gradient Balance

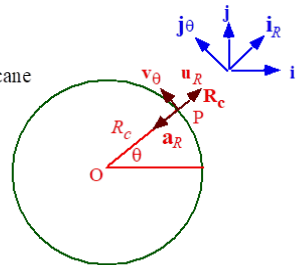
When pressure-gradient force, Coriolis acceleration, and Centrifugal acceleration are in balance - local (inertial) acceleration vanishes, and solving for v_θ,

$$v_\theta = -\frac{fR_c}{2} + \sqrt{\left(\frac{fR_c}{2}\right)^2 + \frac{R_c}{\rho}\frac{\partial p}{\partial R_c}}$$

Example : Low pressure near a tropical cyclone/hurricane

$\frac{\partial p}{\partial R_c} = 45$ hPa/100 km, $\varphi = 15^0$ N, $\rho = 1.06$ kg m^{-3},

$R_c = 70$ km, $p = 850$ hPa, $v_\theta = 52$ m s^{-1}.



Cyclostrophic Balance

$$\frac{du_R}{dt} = fv_\theta + \frac{v_\theta^2}{R_c} - \frac{1}{\rho}\frac{\partial p}{\partial R_c}$$

Cyclostrophic balance : $v_\theta = \sqrt{\frac{R_c}{\rho}\frac{\partial p}{\partial R_c}}$

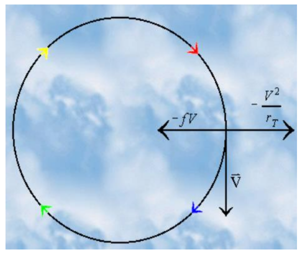
Cyclostrophic balance occurs when the horizontal pressure gradient and centrifugal forces push equally in opposite directions. Smaller vortices, (e.g., tornadoes, dust devils), are cyclostrophically balanced at any latitude.

- The flow must be frictionless, always parallel to the height contours, and the scale of the flow is either small in scale or near the equator, where the coriolis force is essentially zero.

Inertial Balance

- Inertial flow is not one of the more commonly seen flows in the atmosphere, yet it does exist. The inertial wind results from the balance of centrifugal and coriolis force when there is negligible pressure gradient force:

V = -f Rc



- Inertial flows are also known as inertial oscillations, since air parcels under the influence of inertial balance follow circular paths.
- The only situations where inertial flow may be observed are in the center of large anti-cyclones or cyclones, where pressure gradients are very weak.

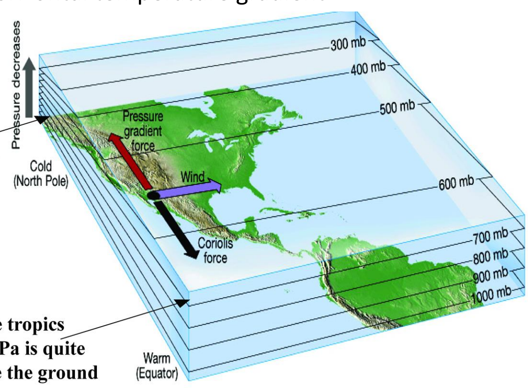
Balanced Flow: Summary

Centrifugal force	PGF	Coriolis force	Balanced flow
	*	*	Geostrophic
*	*		Cyclostrophic
*	*	*	Gradient
*		*	Inertial

Thermal wind

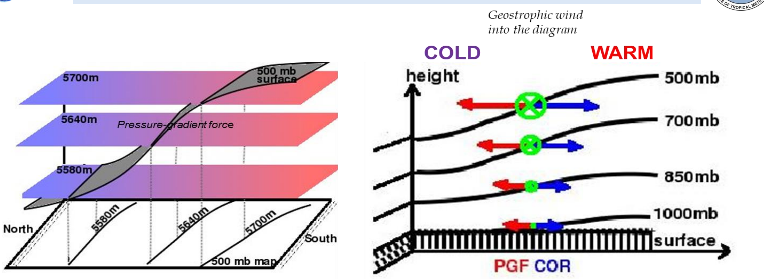
- It results due to vertical shear in the geostrophic wind in the presence of horizontal temperature gradient

700 hPa is quite low near the poles (cold air - dense layer thickness is small)



In the tropics 700 hPa is quite above the ground (warm air - less dense - layer thickness is large)

Thermal wind



A horizontal thermal gradient creates a PGF at upper levels. As you increase in altitude, the pressure gradient between the warm column and the cool column increase. As the PGF increases the magnitude of the wind will increase and so will the Coriolis force (geostrophic winds - looking at an arrow pointing into away from us - into the image).
The vertical change in geostrophic wind is called the geostrophic vertical shear. Since the geostrophic vertical shear is directly proportional to the horizontal temperature gradient, it is also called the Thermal Wind.
Thermal wind is not an actual wind, but a the difference between two winds at different levels.

- **Remember this thumb rule for thermal wind balance:**

Zonal wind shear (vertical) is proportional to (horizontal) meridional TG.

Meridional wind shear (vertical) is proportional to (horizontal) zonal TG.

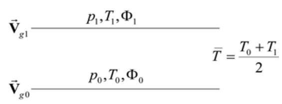
Thermal wind

The magnitude of the thermal wind depends on the horizontal temperature gradient.

An expression for the magnitude of the thermal wind is determined by differentiating the expression of the geostrophic wind with relation to pressure (p)

$$\frac{\partial u_g}{\partial \ln p} = -\frac{1}{f}\frac{\partial}{\partial \ln p}\left(\frac{\partial \Phi}{\partial y}\right) = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial \ln p}\right) = \frac{R_d}{f}\frac{\partial T}{\partial y}$$

$$\frac{\partial v_g}{\partial \ln p} = \frac{1}{f}\frac{\partial}{\partial \ln p}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial \ln p}\right) = -\frac{R_d}{f}\frac{\partial T}{\partial x}$$



using the hydrostatic equation in isobaric system is $\frac{\partial \Phi}{\partial \ln p} = -R_d T$

In vectorial form the thermal wind equation is given by, $\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R_d}{f}\hat{\mathbf{k}} \times \vec{\nabla}_p T$

Integration over p between two pressure levels

$$\vec{V}_T \equiv \vec{V}_g(p_1) - \vec{V}_g(p_0) = -\frac{R_d}{f}\int_{p_0}^{p_1}\left(\hat{\mathbf{k}} \times \vec{\nabla}_p T\right) d\ln p$$

$$\vec{V}_T = \frac{R_d}{f}\ln\left(\frac{p_0}{p_1}\right)\left[-\left(\frac{\partial \bar{T}}{\partial y}\right)_p \hat{\mathbf{i}} + \left(\frac{\partial \bar{T}}{\partial x}\right)_p \hat{\mathbf{j}}\right]$$

$$\vec{V}_T \equiv \frac{1}{f}\hat{\mathbf{k}} \times \vec{\nabla}_p(\Phi_1 - \Phi_0)$$

$$\text{Recall}: \Phi_1 - \Phi_0 = \frac{R_d \bar{T}}{g}\ln\left(\frac{p_0}{p_1}\right)$$

Thermal wind

Thermal wind

$\frac{\partial u}{\partial z} \propto \frac{\partial T}{\partial y}$

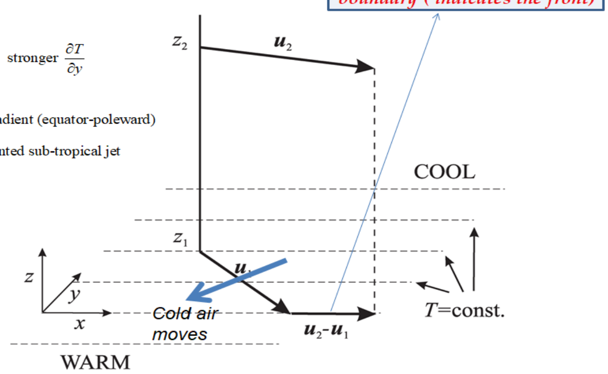
stronger $u_2 - u_1$ ⇔ stronger $\frac{\partial T}{\partial y}$

example:

$\frac{\partial T}{\partial y}$: temperature gradient (equator-poleward)

$u_2 - u_1$: zonally oriented sub-tropical jet

Thermal wind (u_2-u_1) vector separates the cold-warm air boundary (indicates the front)



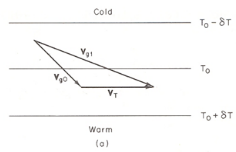
Direction of thermal wind** (u_2 –u_1) **is parallel to isotherms **(Follow Buys-Ballot's law for T: left – cold region, right – warm region) *or thickness lines*.**

Thermal wind

- The thermal wind equation provides useful diagnostic tool for checking analyses of the observed wind and temperature fields for consistency. It can also be used to estimate the mean horizontal temperature advection in a layer

The thermal wind blows parallel to the isotherms (lines of constant temperature / thickness) – with the warm air to the right facing downstream in the NH.

Thus, a geostrophic wind which turns anti-clockwise with height (backs) is associated with cold-air advection ➡



Conversely, clockwise turning (veering) of the geostrophic wind with height implies warm-air advection in the layer ➡

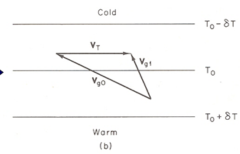


Figure shows the relationship between turning of the geostrophic wind and temperature advection - (a) Backing of wind with height (b) Veering of wind with height

Barotropic and Baroclinic atmosphere

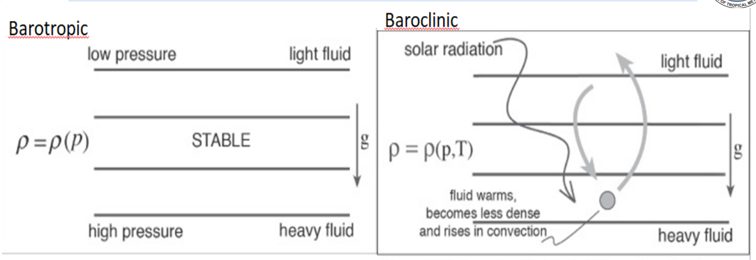


Figure 00.4: Because of the overwhelming importance of gravity, pressure increases downward in the fluid atmosphere and ocean. For gravitational stability, density must also increase downward—as sketched in the diagram—with heavy fluid below and light fluid above. If $\rho = \rho(p)$ only, then the density is independent of T, and the fluid cannot be brought into motion by heating and/or cooling.

Figure 00.5: In contrast to Fig. 4, if depends on both pressure and temperature, $\rho = \rho(p, T)$, then fluid heated by the Sun, for example, can become buoyant and rise in convection. Such a fluid can be energized thermally.

Baroclinic fluid is 'full of life as it converts thermal energy into kinetic energy' (Marshall and Plumb)

Barotropic and Baroclinic atmosphere

- A *barotropic* fluid is one in which surfaces of constant pressure and constant density are parallel (see Fig. 1).

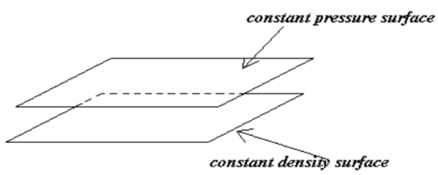


Figure 1: Pressure and density surfaces are parallel in a barotropic fluid.

- In a barotropic fluid the density is constant along a constant pressure surface.
  - From the ideal gas law this implies that, for a barotropic atmosphere, temperature is constant on a constant pressure surface, since $p/\rho = R_d T$, and since both p and ρ are constant on a pressure surface, then so would T.
  - If the atmosphere were truly barotropic there would be no isotherms on a constant pressure map.
- In a barotropic fluid the thermal wind is zero. Therefore, the flow is the same at all levels. ***There is no vertical wind shear in a barotropic atmosphere.***

Barotropic and Baroclinic atmosphere

- As in the Barotropic atmosphere, the density depends only on the pressure, **ρ = ρ(p)**, so that isobaric surfaces are also surfaces of constant density.
- If the atmosphere is Barotropic, then for an ideal gas, the isobaric surfaces will also be isothermal, thus, $\nabla_p . T = 0$
- Consider the thermal wind equation $\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_p T$ and using $\nabla_p . T = 0$, it reduces to $\partial V_g / \partial \ln p = 0$, implying that geostrophic wind is independent of height in a Barotropic atmosphere.
- Thus, Barotropy provides a very strong constraint on the motions in a rotating fluid; i.e. the large-scale motion can depend only on horizontal position and time, not on height.

Barotropic and Baroclinic atmosphere

- If a fluid is not barotropic it is *baroclinic*. In baroclinic fluids the pressure and density surfaces intersect as shown in Fig. 2.
  - In a baroclinic atmosphere there will be a temperature gradient on a constant pressure surface.
  - In a baroclinic atmosphere the flow will be different at different levels.

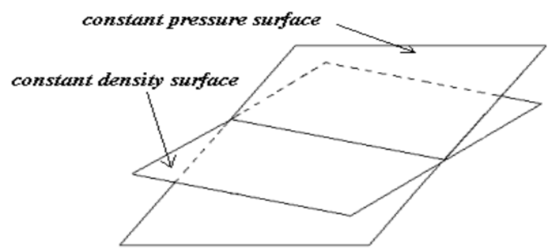


Figure 2: Pressure and density surfaces intersect in a baroclinic fluid.

Barotropic and Baroclinic atmosphere

- An atmosphere in which density depends on both the temperature and the pressure, **ρ = ρ (p, T)**, is referred to as a Baroclinic atmosphere.
- In a Baroclinic atmosphere, the geostrophic wind generally has vertical shear, and this shear is related to the horizontal temperature gradient by the thermal wind equation:

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_p T$$

- Obviously, the Baroclinic atmosphere is of primary importance in dynamic meteorology.
- However, much can be learned by study of the simpler Barotropic atmosphere.

Barotropic and Baroclinic atmosphere

Barotropic

- No temperature gradient on pressure surfaces
- ***no tilt in pressure systems***
- Isobaric surfaces are also the isothermal surfaces - Constant density surfaces lies on isobaric surfaces.
  $$\nabla p \times \nabla \alpha = 0$$
- Geostrophic winds are independent of height – **no thermal wind.**
- ***Pressure gradients – important in barotropic atmosphere – more ideal for tropics.***

Baroclinic

- Temperature gradients exists on pressure surfaces - valid for mid-latitudes/high latitudes
- **Thermal wind exists – *pressure systems tilt with height***
- Geostrophic winds change with height (veering/backing)
- Constant density surfaces intersect the isobaric surfaces. That is,
  $$\nabla p \times \nabla \alpha \neq 0$$
- ***Temperature gradients – important in baroclinic atmosphere – more ideal in mid-latitudes.***

Circulation and Vorticity

Measurement of Rotation

- Circulation and vorticity are the two primary measures of rotation in a fluid.
- Circulation, which is a scalar integral quantity, is a *macroscopic* measure of rotation for a finite area of the fluid.
- Vorticity, however, is a vector field that gives a *microscopic* measure of the rotation at any point in the fluid.

Circulation and Vorticity

Circulation theorems and application

Circulation:

Definition:

Circulation is defined as a macro-scale measure of rotation of fluid. Mathematically it is defined as a line integral of the velocity vector around a closed path, about which the circulation is measured.

Circulation may be defined for an arbitrary vector field, say, $\vec{B}$. Circulation 'C_B' of an arbitrary vector field $\vec{B}$ around a closed path, is mathematically expressed as a line integral of $\vec{B}$ around that closed path, i.e., $C_B = \oint \vec{B}.d\vec{l}$.

In Meteorology, by the term, 'Circulation' we understand the circulation of velocity vector. Hence, in Meteorology circulation around a closed path is given by C = $\oint \vec{V} . \vec{dl}$ ....(C1.1). From this expression it is clear that circulation is a scalar quantity.

Circulation and Vorticity

Circulation theorems and application

Conventionally, sign of circulation is taken as positive (or negative) for an anticlockwise rotation (or for a clockwise rotation) in the Northern hemisphere. Sign convention is just opposite in the Southern hemisphere. Since we talk about absolute and relative motion, hence we can talk about absolute circulation and relative circulation. They are respectively denoted by C_a and C_r respectively and are defined as follows:

$$C_a = \oint \vec{V_a} . \vec{dl} \quad .... (C1.2)$$

and $$C_r = \oint \vec{V_r} . \vec{dl} \quad .... (C1.3)$$

Where $\vec{V_a}$ and $\vec{V_r}$ are the absolute and relative velocities respectively.

Circulation and Vorticity

Circulation theorems and application

Stokes Theorem:-

It states that the line integral of any vector $\vec{B}$ around a closed path is equal to the surface integral of $\vec{\nabla} \times \vec{B}.\hat{n}$ over the surface 'S' enclosed by the closed path, where $\hat{n}$ is the outward drawn unit normal vector to the surface 'S'.

So, $\oint \vec{B} \cdot \vec{dl} = \iint (\vec{\nabla} \times \vec{B}) \cdot \hat{n}\, ds$.

***The Circulation Theorems*:**

Circulation theorems deal with the change in circulation and its cause(s).

For an arbitrary vector field, $\vec{B}$ the circulation theorem states that the time rate of change of circulation of $\vec{B}$ is equal to the circulation of the time rate of change of $\vec{B}$, i.e.,

$$\frac{d}{dt}\oint \vec{B}.d\,l = \oint \frac{d\vec{B}}{dt}.\vec{dl} \quad ....(C1.4)$$

This theorem may be applied to the absolute velocity vector ($\vec{V}_a$) as well as to the relative velocity vector ($\vec{V}_r$).

Circulation and Vorticity

Circulation theorems and application

Kelvin's Circulation theorem:

It is the circulation theorem, when applied to the absolute velocity ($\vec{V}_a$) of fluid motion.

So according to Kelvin's Circulation theorem,

$$\frac{d_a C_a}{dt} = \oint \frac{d_a \vec{V_a}}{dt} . \vec{dl} \quad .....(C1.5).$$

Proof: We know that $C_a = \oint \vec{V_a} . \vec{dl}$

So, $\frac{d_a C_a}{dt} = \frac{d_a}{dt} \oint \vec{V}_a . d\vec{l}$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a \vec{V}_a}{dt} . \vec{dl} + \oint \vec{V}_a . \frac{d_a}{dt}(\vec{dl})$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a \vec{V}_a}{dt} . \vec{dl} + \oint \vec{V}_a . d_a \vec{V}_a$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a \vec{V}_a}{dt} . \vec{dl}$

Conventionally, $\frac{d_a C_a}{dt}$ or $\frac{dC_r}{dt}$ are known as acceleration of circulation (absolute or relative).

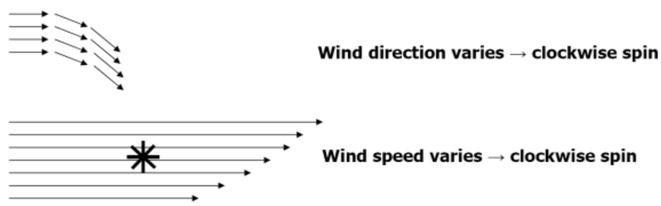
So, in Meteorology, circulation theorem simply states that the acceleration of circulation is equal to the circulation of acceleration.

Circulation and Vorticity

Vorticity

Vorticity is the microscopic measure of spin and rotation in a fluid.

Vorticity is defined as the curl of the velocity: $\nabla \times \vec{V}$



Absolute vorticity (inertial reference frame): $\vec{\omega}_a \equiv \nabla \times \vec{V}_a$

Relative vorticity (relative to rotating earth): $\vec{\omega} \equiv \nabla \times \vec{V}$

Circulation and Vorticity

Expansion of relative vorticity into Cartesian components:

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

For large scale dynamics, the vertical component of vorticity is most important. The vertical components of absolute and relative vorticity in vector notation are:

$\zeta = \hat{k} \cdot \left(\nabla \times \vec{V}\right)$ relative vorticity

$\eta = \hat{k} \cdot \left(\nabla \times \vec{V}_a\right)$ absolute vorticity

From now on, vorticity implies the vertical component (unless otherwise stated.)

Circulation and Vorticity

The absolute vorticity is equal to the relative vorticity plus the earth's vorticity. Since the earth's vorticity is

$$\hat{k} \cdot \left(\nabla \times \vec{V}_e\right) = 2\Omega \sin\phi = f$$

then

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f = \zeta + f$$

For large scale circulations, a typical magnitude for vorticity is

$$\zeta \approx \frac{U}{L} = 10^{-5}\ s^{-1}$$

End of Lecture - 03

Dynamic Meteorology

Lecture - 04

[Tuesday, 2022-XX-XX]

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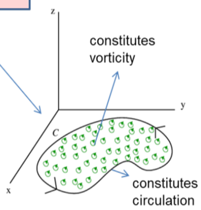
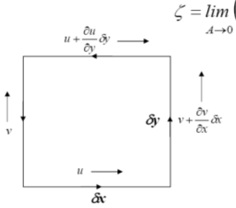
Circulation and Vorticity

Vorticity=Circulation per unit area
Circulation (scalar)
Vorticity (vector)



The relationship between relative vorticity ζ and circulation can be seen by considering the following expression, in which we will define the relative vorticity as the circulation about a closed contour in the horizontal plane divided by the area enclosed by that contour, in the limit as the area approaches zero.

$$\zeta = \lim_{A\to 0}\left(\oint \vec{V}\cdot d\vec{l}\right)A^{-1}$$



Evaluating $\vec{V}\cdot d\vec{l}$ for each side of the rectangle yields the circulation:

$$\delta C = u\,\delta x + \left(v+\frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u+\frac{\partial u}{\partial y}\delta y\right)\delta x - v\,\delta y$$

$$\delta C = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\delta x\,\delta y \rightarrow \frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = \zeta$$

Circulation and Vorticity

Vorticity equation

Vorticity=Circulation per unit area
Circulation (scalar)
Vorticity (vector)

Circulation $C = \varsigma A$ (macroscopic rotation of fluid)

Relative vorticity $\varsigma = \hat{\mathbf{k}}\cdot\nabla\times\vec{\mathbf{V}}$ (spin of a fluid - microscopic) $= \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

Planetary vorticity $= f = 2\Omega\sin\varphi$

Absolute vorticity $= \varsigma + f = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} + f$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv+\alpha\frac{\partial p}{\partial x}\right)=0 \quad (1)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu+\alpha\frac{\partial p}{\partial y}\right)=0 \quad (2)$$

◀Differentiating the zonal and meridional momentum equations

Subtracting (1) from (2),

$$\frac{\partial\varsigma}{\partial t}+\vec{\mathbf{V}}\cdot\vec{\nabla}\varsigma+v\frac{\partial f}{\partial y} = -(\varsigma+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\frac{d}{dt}(\varsigma+f) = -(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) - \hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

A B C D

For synoptic scale motions, B > (C, D)

A=rate of change of absolute vorticity following the fluid motion
B=Effect of horizontal velocity divergence on vorticity
C=Transfer of vorticity between horizontal and vertical components (twisting or tilting term)
D=Baroclinicity (solenoidal term)

Circulation and Vorticity

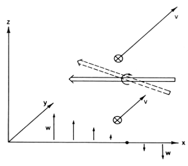
$$\frac{d}{dt}(\varsigma+f) = -(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) - \hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

- The equation above states that the rate of change of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence term, the tilting or twisting term, and the solenoidal term, respectively.
- The concentration or dilution of vorticity by the divergence field (the first term on the right) is the fluid analog of the change in angular velocity resulting from a change in the moment of inertia of a solid body when angular momentum is conserved.
- If the horizontal flow is divergent, the area enclosed by a chain of fluid parcels will increase with time and if circulation is to be conserved, the average absolute vorticity of the enclosed fluid must decrease (i.e., the vorticity will be diluted).

Circulation and Vorticity

$$\frac{d}{dt}(\varsigma+f) = -(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) - \hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

- If, however, the flow is convergent, the area enclosed by a chain of fluid parcels will decrease with time and the vorticity will be concentrated. This mechanism for changing vorticity following the motion is very important in synoptic-scale disturbances.
- The second term represents vertical vorticity generated by the tilting of horizontally oriented components of vorticity into the vertical by a non-uniform vertical motion field.
- The figure shows a region where the **y** component of velocity is increasing with height so that there is a component of shear vorticity oriented in the negative **x** direction as indicated by the double arrow.



Circulation and Vorticity

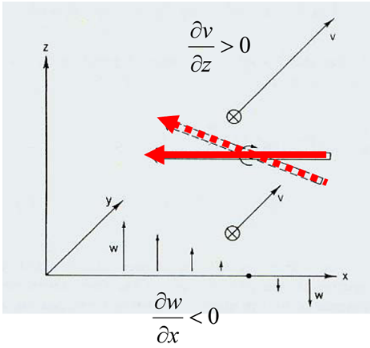
$$\frac{d}{dt}(\varsigma+f)=-(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}}-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)-\hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

- If at the same time there is a vertical motion field in which **w** decreases with increasing **x**, advection by the vertical motion will tend to tilt the vorticity vector initially oriented parallel to **x** so that it has a component in the vertical. Thus, if **∂v/∂z > 0** and **∂w/∂x < 0**, there will be a generation of positive vertical vorticity.

- Finally, the third term on the right is just the microscopic equivalent of the solenoidal term in the circulation theorem.

Circulation and Vorticity

Vorticity equation – Tilting term



In this example, vertical shear of v-component wind is producing shear vorticity about an east-west axis. The orientation of the vorticity vector is shown by the solid red arrow.

East-west variations in the vertical velocity twist or tilt this "vortex tube" toward a more vertical orientation, as indicated by the broken red arrow. This gives the vorticity vector a component in the z-direction, indicating a transfer of vorticity from the horizontal to the vertical.

$$\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}<0\rightarrow\frac{d(\zeta+f)}{dt}>0$$

$$\frac{d_H}{dt}(\varsigma+f)=-(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}}-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)-\hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

Circulation and Vorticity

Vorticity equation – Tilting/twisting term

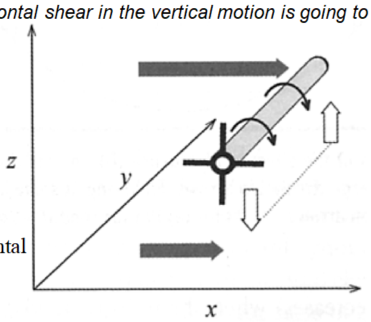
Vertical shear in the horizontal motion is going to twist
Horizontal shear in the vertical motion is going to tilt

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial w}{\partial y}>0,\quad\frac{\partial u}{\partial z}>0$$

$$\frac{d}{dt}(\varsigma+f)>0$$

twisting **arises** due to vertical shear in the horizontal motion, and horizontal shear in the vertical motion



$$\frac{d_H}{dt}(\varsigma+f)=-(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}}-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)-\hat{\mathbf{k}}\cdot(\nabla\alpha\times\nabla p)$$

Circulation and Vorticity

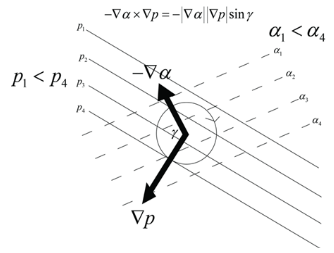
Solenoidal/baroclinicity

We can write **Bjerknes circulation theorem** for a baroclinic atmosphere as,

$$\frac{D\Gamma_{circulation}}{Dt}=-\int_A(\nabla\alpha\times\nabla p)\cdot d\vec{\mathbf{A}}=-\int_A|\nabla\alpha||\nabla p|\sin\gamma\, dA$$

In this figure, $\frac{D\Gamma_{circulation}}{Dt}<0\ \Rightarrow$ clockwise circulation would develop.

In general, the circulation that develops would be such that density and pressure surfaces would become parallel (baroclinic $\rightarrow$ barotropic)



Scale Analysis of Vorticity Equation

∂ζ/∂t + u*∂ζ/∂x + v*∂ζ/∂y + w*∂ζ/∂z + (ζ + f)*(∂u/∂x + ∂v/∂y) + (∂w/∂x*∂v/∂z − ∂w/∂y*∂u/∂z) + v*df/dy = 1/ρ²*(∂ρ/∂x * ∂p/∂y − ∂ρ/∂y * ∂p/∂x)

- Consider a mid-latitude synoptic system with scale as follows:

$U\sim 10\ \mathrm{m\ s^{-1}}$	horizontal scale
$W\sim 1\ \mathrm{cm\ s^{-1}}$	vertical scale
$L\sim 10^6\ \mathrm{m}$	length scale
$H\sim 10^4\ \mathrm{m}$	depth scale
$\delta p\sim 10\ \mathrm{hPa}$	horizontal pressure scale
$\rho\sim 1\ \mathrm{kg\ m^{-3}}$	mean density
$\delta\rho/\rho\sim 10^{-2}$	fractional density fluctuation
$L/U\sim 10^5\ \mathrm{s}$	time scale
$f_0\sim 10^{-4}\ \mathrm{s^{-1}}$	Coriolis parameter
$\beta\sim 10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$	"beta" parameter

Therefore:

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\lesssim\frac{U}{L}\sim 10^{-5}\mathrm{s}^{-1}$$

and:

$$\zeta/f_0\lesssim U/(f_0L)\equiv\mathrm{Ro}\sim 10^{-1}$$

- For mid-latitude synoptic-scale systems, the relative vorticity (**ζ**) is often small compared to the planetary vorticity (**f**) where **ζ** may be neglected compared to **f** in the divergence term in the vorticity equation:

$$(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\approx f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Scale Analysis of Vorticity Equation

∂ζ/∂t + u*∂ζ/∂x + v*∂ζ/∂y + w*∂ζ/∂z + (ζ + f)*(∂u/∂x + ∂v/∂y) + (∂w/∂x*∂v/∂z − ∂w/∂y*∂u/∂z) + v*df/dy = 1/ρ²*(∂ρ/∂x * ∂p/∂y − ∂ρ/∂y * ∂p/∂x)

- The magnitudes of the various terms in the equation above can be estimated as:

$$\frac{\partial\zeta}{\partial t},\ u\frac{\partial\zeta}{\partial x},\ v\frac{\partial\zeta}{\partial y}\sim\frac{U^2}{L^2}\sim 10^{-10}\ \mathrm{s^{-2}}$$

$$w\frac{\partial\zeta}{\partial z}\sim\frac{WU}{HL}\sim 10^{-11}\ \mathrm{s^{-2}}$$

$$v\frac{df}{dy}\sim U\beta\sim 10^{-10}\ \mathrm{s^{-2}}$$

$$f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\lesssim\frac{f_0U}{L}\sim 10^{-9}\ \mathrm{s^{-2}}$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)\lesssim\frac{WU}{HL}\sim 10^{-11}\ \mathrm{s^{-2}}$$

$$\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)\lesssim\frac{\delta\rho\delta p}{\rho^2L^2}\sim 10^{-11}\ \mathrm{s^{-2}}$$

Scale Analysis of Vorticity Equation

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Scaled for mid-latitude synoptic-scale weather

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical scale
$L \sim 10^6\,\mathrm{m}$	length scale
$H \sim 10^4\,\mathrm{m}$	depth scale
$\delta p \sim 10\,\mathrm{hPa}$	horizontal pressure scale
$\rho \sim 1\,\mathrm{kg\,m^{-3}}$	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
$L/U \sim 10^5\,\mathrm{s}$	time scale
$f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$	Coriolis parameter
$\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$	"beta" parameter

$$\frac{D_h(\zeta + f)}{Dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad \frac{D_h}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

In intense cyclonic storms, the relative vorticity should be retained in the divergence term.

Natural Coordinates

- The natural coordinate system is defined by the orthogonal set of unit vectors **t**, **n**, and **k**.
- Unit vector **t** is oriented parallel to the horizontal velocity at each point; unit vector **n** is normal to the horizontal velocity and is oriented so that it is positive to the left of the flow direction; and unit vector **k** is directed vertically upward.
- In this system the horizontal velocity may be written **V** = *V* **t** where V , the horizontal speed, is a nonnegative scalar defined by *V* ≡ Ds/Dt, where s(x, y, t) is the distance along the curve followed by a parcel moving in the horizontal plane.
- The acceleration following the motion is thus: $\frac{D\mathbf{V}}{Dt} = \frac{D(V\mathbf{t})}{Dt} = \mathbf{t}\frac{DV}{Dt} + V\frac{D\mathbf{t}}{Dt}$

Natural Coordinates

- The rate of change of **t** following the motion may be derived from geometrical considerations with the aid of the following figure:

$$\delta\psi = \frac{\delta s}{|R|} = \frac{|\delta\mathbf{t}|}{|\mathbf{t}|} = |\delta\mathbf{t}|$$

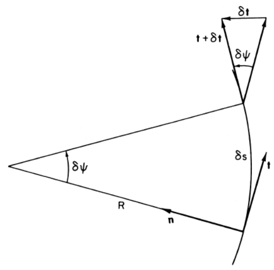


Fig. 3.1 Rate of change of the unit tangent vector **t** following the motion.

- Here **R** is the radius of curvature following the parcel motion, and we have used the fact that |t| = 1. By convention, **R** is taken to be positive when the center of curvature is in the positive **n** direction.
- Thus, for **R** > 0, the air parcels turn toward the left following the motion, and for **R** < 0 the air parcels turn toward the right following the motion.

Natural Coordinates

- Noting that in the limit **δs** → 0, **δt** is directed parallel to **n**, the above relationship yields **Dt/Ds = n/R**. Thus,

$$\frac{D\mathbf{t}}{Dt} = \frac{D\mathbf{t}}{Ds}\frac{Ds}{Dt} = \frac{\mathbf{n}}{R}V$$

and

$$\frac{D\mathbf{V}}{Dt} = \mathbf{t}\frac{DV}{Dt} + \mathbf{n}\frac{V^2}{R}$$

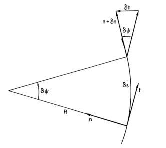


Fig. 3.1 Rate of change of the unit tangent vector t following the motion.

- Therefore, the acceleration following the motion is the sum of the rate of change of speed of the air parcel and its centripetal acceleration due to the curvature of the trajectory.
- Because the Coriolis force always acts normal to the direction of motion, its natural coordinate form is simply: $-f\mathbf{k}\times\mathbf{V} = -fV\mathbf{n}$
- Whereas the pressure gradient force can be expressed as:

$$-\nabla_p\Phi = -\left(\mathbf{t}\frac{\partial\Phi}{\partial s} + \mathbf{n}\frac{\partial\Phi}{\partial n}\right)$$

Natural Coordinates

- The horizontal momentum equation may thus be expanded into the following component equations in the natural coordinate system:

$$\frac{DV}{Dt} = -\frac{\partial\Phi}{\partial s}$$

$$\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n}$$

- These two equation express the force balances parallel to and normal to the direction of flow, respectively.
- For motion parallel to the geopotential height contours, **∂ϕ/∂s** = 0 and the speed is constant following the motion.
- If, in addition, the geopotential gradient normal to the direction of motion is constant along a trajectory, the second equation also implies that the radius of curvature of the trajectory is also constant.
- In that case, the flow can be classified into several simple categories depending on the relative contributions of the three terms in the second equation to the net force balance.

Potential vorticity

To understand the concept of Potential vorticity, first we may refer to the popular circus play, where a girl is standing at the centre of a rotating disc. As the girl stretches her arm, the disc rotates at a slower rate and as she withdraws her arms the disc rotates at a faster rate. Generally this example is referred in solid rotation to illustrate the conservation of angular momentum. This example hints us to search a quantity in the fluid rotation, which is analogous to the angular momentum in solid rotation.

For that we consider an air column of unit radius. Now, consider that the air column shrinks down i.e. its depth decreases. As it shrinks down, its radius increases and then as per the above example column will rotate at a slower speed. Also if the air column stretches vertically i.e. if its depth increases, then its radius decreases and rate of rotation increases

Potential vorticity

So, it's clear that the rate of rotation of the air column increases or decreases as its depth increases or decreases.

Thus for a rotating air column, we can say that the rate of rotation is proportional to the depth of the air column.

Now for fluid motion rate of rotation and vorticity are analogous

$\therefore$ Vorticity $\propto$ Depth

Vorticity/Depth = constant

Thus in the fluid rotation the quantity (Vorticity/Depth) remains constant as in the solid rotation angular momentum remains constant. So this quantity is analogous to the angular momentum. It is known as potential vorticity.

Potential vorticity

A model that has proved useful for elucidating some aspects of the horizontal structure of large-scale atmospheric motions is the barotropic model. In the most general version of this model, the atmosphere is represented as a homogeneous incompressible fluid of variable depth, $h(x, y, t) = z_2 - z_1$, where z_2 and z_1 are the heights of the upper and lower boundaries, respectively.

In a barotropic (incompressible) fluid, the vorticity equation (combined with the continuity equation) for can be written as:

$$\frac{d}{dt}(\varsigma+f) = -(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_H = (\varsigma+f)\frac{\partial w}{\partial z} = (\varsigma+f)\frac{w(z_2)-w(z_1)}{h}$$

$$h\frac{d}{dt}(\varsigma+f) = (\varsigma+f)\left[\frac{dz_2}{dt}-\frac{dz_1}{dt}\right] = (\varsigma+f)\frac{dh}{dt}$$

$$\frac{1}{(\varsigma+f)}\frac{d}{dt}(\varsigma+f) = \frac{1}{h}\frac{dh}{dt} \Rightarrow \frac{d}{dt}\ln(\varsigma+f) = \frac{d\ln h}{dt} \Rightarrow \frac{d}{dt}\ln\left(\frac{\varsigma+f}{h}\right) = 0$$

$\frac{d}{dt}\ln\left(\frac{\varsigma+f}{h}\right) = 0$ implies that $\frac{d}{dt}\left(\frac{\varsigma+f}{h}\right) = 0$ BAROTROPIC VORTICITY EQUATION

$\eta \equiv \frac{\varsigma+f}{h} =$ potential vorticity $\Rightarrow$ potential vorticity is conserved following the motion in a barotropic atmosphere. This is also called '*Rossby potential vorticity*'.

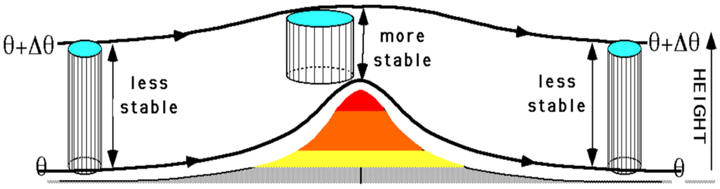
Potential vorticity

The real atmosphere is usually baroclinic, so it is more appropriate to use conservations of Ertel's potential vorticity.

$$P = -g(\varsigma+f)\frac{\partial\theta}{\partial p}$$

In this case, the top of the atmosphere is not constrained to be a rigid lid, but instead we look at a column of atmosphere constrained between two isentropes (surfaces of constant potential temperature). In this case the quantity P must remain constant following the fluid column.

As the flow approaches the hill, both the bottom isentrope and the top isentrope deflect upwards, but the top deflects to a much lesser extent (although it begins its deflection prior to the bottom deflecting). The diagram would look something like that below.



Exercises

1. **What would be length and time scales when you perform scale analysis for the these three specific phenomena:**

SYNOPTIC SCALE
extra-tropical cyclone

MESOSCALE
Sea-breeze

MICROSCALE
Cumulus cloud

2. **Whether cyclone formation takes place near the equator, why?**

3. **Can the Coriolis force change the magnitude of the wind velocity?**

4. **In general why wind does not move upward despite a strong pressure gradient in the vertical direction?**

5. **Write the momentum equation of a body of mass m moving with a variable velocity "V" in a rotating coordinate system**

Thank You!

End of Lecture - 04